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Nonlinear ripples of Kelvin–Helmholtz type which arise from an interfacial mode interaction

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An analysis is made of the small-amplitude capillary–gravity waves which occur on the interface of two fluids and which arise out of the interaction between the Mth and Nth harmonics of the fundamental mode. The method employed is that of multiple scales in both space and time and a pair of coupled nonlinear partial differential equations for the slowly varying wave amplitudes is derived. These equations describe, correct up to third order, the progression of a wavetrain and are generalizations of the nonlinear Schrödinger-type equations used by many authors to model wave propagation. The equations are solved and formal power series expansions of the corresponding wave profiles obtained. Many different wave configurations can arise, some symmetric others asymmetric. It is found that an important influence on the type of waves which can occur is whether the ratio of the interacting wave modes is greater or less than two. Finally, an examination of the stability of the waves to plane wave perturbations is carried out.

1. Introduction

Our purpose in this paper is to make an investigation into the resonant capillary–gravity waves of Kelvin–Helmholtz type which may arise on the interface of two semi-infinite stratified fluids which are moving with uniform velocities parallel to their interface. We shall be concerned with those ripples which are caused by the interaction of the Mth and Nth harmonics of a fundamental mode. (Here M is greater than N and for technical reasons to be discussed later it is assumed that M is not equal to $2N$ or $3N$.) We shall seek small-amplitude travelling waves and will approach the problem by means of a weakly nonlinear analysis. Our method is to introduce a small parameter ϵ, representing the wave steepness, and then to expand the velocity potentials and wave profile in ascending powers of ϵ. By assuming the wave amplitudes are slowly varying in both space and time, we are able to derive a pair of coupled nonlinear partial differential equations which model the evolution of the waves, correct up to third order. It turns out that there are four significant parameters in the problem: M, N, V and ρ. Here V is a measure of the relative velocities of the fluids and ρ is the ratio of their densities. Solutions to the equations will be found which correspond to Stokes-type travelling waves and depictions of the wave profiles are presented. It will be seen that in general, for fixed values of the parameters, a large variety of wave profiles is possible and both symmetric and asymmetric waves may occur. However, there are certain values of the parameters for which only one type of wave is to be found. Having obtained the wave profiles we then proceed to consider their stability to plane wave perturbations which may be at any angle to the direction of propagation. Results will be presented which show how the stability is affected by changes in the velocity and density parameters (for fixed values of the wave harmonics) and also by changes in the values of M and N (for fixed values of the remaining parameters).

The topic of mode interactions between different harmonics of water waves in an open channel (and indeed in other physical systems) has received much attention during this century. A seminal paper is that of Wilton (1915) who drew attention to the fact that a wave and its second harmonic may travel at the same speed, thereby producing a resonant interaction. This work was extended somewhat by Pierson & Fife (1961) who obtained results for deep-water Wilton ripples valid up to second order and Nayfeh (1970) who considered finite-depth waves and produced results valid to third order. Later Nayfeh (1971) went on to consider third-order resonances, but the first quite general analysis of resonant capillary–gravity waves was that undertaken by Chen & Saffman (1979). They undertook a formal analysis of the waves formed by interactions between the Mth and Nth harmonics and obtained a comprehensive description of the different wave configurations which are possible. Later, Toland & Jones (1985) and Jones & Toland (1986) performed a rigorous mathematical analysis of the problem by transforming it into an integral equation and using the techniques of modern functional analysis and bifurcation theory. In this way they vindicated most of Chen & Saffman's conclusions. Their work was extended somewhat by Aston (1991) who considered the case when the interaction takes place between the fundamental and one of its higher harmonics. All the studies mentioned so far have been, to a greater or lesser extent, analytic but there have been some purely numerical studies of the capillary–gravity wave problem. Chen & Saffman (1980) used numerical techniques to investigate finite-amplitude capillary–gravity waves and a similar investigation was made by Schwartz & Vanden-Broeck (1979). Much later Aston (1993) used numerical methods to provide an extensive description of a very large number of travelling wave solutions to the capillary–gravity wave mode interaction problem, including some new branches which are not connected to the trivial solution.

The stability of waves in an open channel has also been the subject of extensive study over the years although it is only comparatively recently that attention has been focused on the stability of resonant waves. Much work on wave stability has stemmed from the pioneering research of Benjamin & Feir (1967) who showed how a progressive gravity wavetrain is unstable in the presence of sidebands. However it was not until considerably later, in the work of Bridges & Mielke (1995), that a rigorous proof of this instability was given. Another result of major significance in this area is that of Zakharov (1968) who was the first to show that the evolution of a wavetrain may be described up to third order by a nonlinear Schrödinger equation. Zakharov's work dealt with gravity waves on deep water and his results have been the subject of many generalizations. Davey & Stewartson (1974) considered gravity waves in a channel of finite depth while Djordjevic & Redekopp (1977) took surface tension effects into account. Jones (1992, 1993, 1994*a*) derived the equations modelling the situation for various different types of resonance and in addition considered the stability of the waves which arise in these cases.

The studies mentioned above all dealt with flow in an open channel and subject to constant atmospheric pressure. However flows in stratified fluids have, of course, attracted the attention of many researchers over the years. One of the best known and most important results is the classical Kelvin–Helmholtz instability (see, for instance Chandrasekhar 1961 or Craik 1985). A comprehensive investigation into

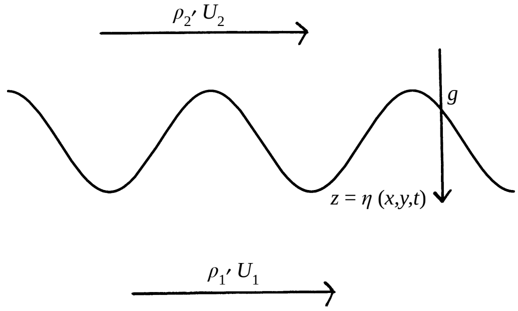

FIGURE 1. The flow configuration.

the stability of non-resonant interfacial waves between two bounded fluids was made by Christodoulides & Dias (1995). They used both variational techniques and the method of multiple scales to attack the problem and sought both travelling waves and standing waves. Nayfeh & Saric (1972) used the method of multiple scales to investigate the nonlinear stability of a wavetrain at the interface of two fluids. Mostly, their work was concerned with non-resonant waves but they did include some results on second-harmonic resonance. The instability of second-harmonic resonant interfacial waves has also been the subject of recent studies by Bontozoglou & Hanratty (1990) and Christodoulides & Dias (1994). Both of these studies made use of a Lagrangian formulation based on that of Miles (1986*a, b*) and employed a combination of analytic and numerical techniques.

2. Problem formulation

Under consideration is the irrotational motion of two semi-infinite stratified inviscid incompressible fluids. We introduce a three-dimensional Cartesian coordinate system so when the motion is undisturbed the interface of the fluids is given by the plane $z = 0$ and gravity acts in the negative z-direction. We assume that the lower fluid (that occupying $z \leqslant 0$) has density ρ_1, while the upper fluid (that occupying $z \geqslant 0$) has density ρ_2, where $\rho_2 \leqslant \rho_1$ so the lighter fluid is on top. We further assume that when the motion is undisturbed the fluids are moving in the x-direction with uniform horizontal velocities U_1 and U_2 (see figure 1).

Then since the motion is supposed to be irrotational we may introduce potential functions $\phi_j(x, y, z, t)$ $(j = 1, 2)$ describing the perturbed flow so that the total potential for the motion is

$$\phi'_j(x, y, z, t) = U_j x + \phi_j(x, y, z, t).$$

We further introduce a function $\eta(x, y, t)$ so that the interface of the fluids is given by $z = \eta(x, y, t)$. Then the equations which describe the motion are

$$\nabla^2\phi_1 = 0, \qquad z \leqslant \eta, \quad \nabla^2\phi_2 = 0, \qquad z \geqslant \eta, \tag{2.1a,b}$$

$$\phi_1 \to 0, \qquad z \to -\infty; \quad \phi_2 \to 0, \qquad z \to \infty, \tag{2.1c,d}$$

$$\eta_t - \phi_{jz} + U_j\eta_x + \phi_{jx}\eta_x + \phi_{jy}\eta_y = 0, \qquad z = \eta, \qquad j = 1, 2, \tag{2.1e}$$

$$\rho\phi_{2t} - \phi_{1t} + \rho U_2\phi_{2x} - U_1\phi_{1x} + (\rho - 1)g\eta + \tfrac{1}{2}\rho(\phi_{2x}^2 + \phi_{2y}^2 + \phi_{2z}^2) - \tfrac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2 + \phi_{1z}^2)$$
$$+ \frac{S}{\rho_1}\frac{(\eta_{xx}(1 + \eta_y^2) + \eta_{yy}(1 + \eta_x^2) - 2\eta_x\eta_y\eta_{xy})}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} = 0, \qquad z = \eta. \tag{2.1f}$$

Here S denotes surface tension, g the force of gravity and $\rho = \rho_2/\rho_1$. The conditions (2.1e) are the usual kinematic conditions on the interface and (2.1f) arises from applying Bernoulli's condition there.

We are interested in small-amplitude sinusoidal disturbances which are formed by the interaction of the Mth and Nth harmonics of a fundamental mode. As a first step we linearize the boundary conditions (2.1e, f) about the zero solution to obtain

$$\eta_t - \phi_{jz} + U_j \eta_x = 0, \quad z = 0, \tag{2.2a}$$

$$\rho\phi_{2t} - \phi_{1t} + \rho U_2 \phi_{2x} - U_1 \phi_{1x} + (\rho - 1) g\eta + \frac{S}{\rho_1}(\eta_{xx} + \eta_{yy}) = 0, \quad z = 0. \tag{2.2b}$$

We shall seek solutions to these equations which are independent of y, so that to a first approximation the perturbed flow is in the same direction as the laminar flow. It is then a routine exercise to verify that for any distinct positive integers M and N, the functions

$$\phi_1 = \frac{\mathrm{i}}{k}(U_1 k - \omega)\mathrm{e}^{\mathrm{i}n(kx - \omega t) + nkz}, \tag{2.3a}$$

$$\phi_2 = -\frac{\mathrm{i}}{k}(U_2 k - \omega)\mathrm{e}^{\mathrm{i}n(kx - \omega t) - nkz}, \tag{2.3b}$$

$$\eta = \mathrm{e}^{\mathrm{i}n(kx - \omega t)}, \quad n = M, N \tag{2.3c}$$

are solutions to (2.2) provided

$$\frac{S}{\rho_1} = \frac{(1-\rho)g}{MNk^2} \tag{2.4a}$$

and

$$\rho(U_2 k - \omega)^2 + (U_1 k - \omega)^2 = \frac{gk(1-\rho)(M+N)}{MN}. \tag{2.4b}$$

Equation (2.4b) has the solutions

$$\omega = \frac{kMN(U_1 + \rho U_2) \pm [MN(M+N)gk(1-\rho^2) - M^2N^2\rho k^2(U_1 - U_2)^2]^{1/2}}{MN(\rho + 1)}, \tag{2.4c}$$

and hence the waves are stable if the values of ω are real. Thus the flow is stable when the term under the square root sign is positive, i.e. provided

$$(U_1 - U_2)^2 < \frac{g}{\rho k}(1-\rho^2)\left(\frac{1}{M} + \frac{1}{N}\right). \tag{2.4d}$$

This is the classical result of linear Kelvin–Helmholtz instability theory. Henceforth we assume that condition (2.4d) (as well as (2.4a, b)) is satisfied. This should not restrict the values of U_1 and U_2 because of the right-hand side of (2.4d) can be made arbitrarily large by choosing k, the wavenumber, to be sufficiently small. In any case, as we shall see, it is the ratio of the velocities which is the most significant parameter. (It might be worth remarking that a further avenue for research would be a study of the waves which occur when the parameters are perturbed slightly from their critical values.) Our aim is now to develop a weakly nonlinear theory for a wave whose wavenumber, frequency and amplitude are all temporal and spacial slowly varying functions. To facilitate this we introduce a parameter ϵ which satisfies $|\epsilon| \ll 1$ and represents the order of magnitude of the steepness of the wave. To simplify the algebra we scale the length coordinate so that the wavenumber is equal to unity. We also introduce the slow variables $X = \epsilon x, Y = \epsilon y, T = \epsilon t, T_1 = \epsilon^2 t$ and in addition set $E(n) = \mathrm{e}^{\mathrm{i}n(x - \omega t)}$ for any $n \geqslant 1$ and put $V_j = U_j - \omega$ for $j = 1, 2$.

It is our ultimate intention to derive a pair of nonlinear equations which model up to cubic order the progression of a weakly nonlinear wave caused by the interaction of the Mth and Nth harmonic of the fundamental. To achieve this it is necessary to develop $\phi_j (j = 1, 2)$ and η in ascending powers of ϵ. Retaining relevant terms and bearing in mind (2.1a, b), it follows that

$$\begin{aligned}\phi_1 = \epsilon \Big[& \mathrm{i}V_1 C_N + \epsilon(A_N^{(2)} + zV_1 C_{NX}) \\ & + \epsilon^2 (A_N^{(3)} - \mathrm{i}zA_{NX}^{(2)} - \frac{\mathrm{i}zV_1}{2N} C_{NYY} - \frac{\mathrm{i}z^2}{2} V_1 C_{NXX}) \Big] E(N)\mathrm{e}^{Nz} \\ & + \epsilon \Big[\mathrm{i}V_1 C_M + \epsilon(A_M^{(2)} + zV_1 C_{MX}) \\ & + \epsilon^2 (A_M^{(3)} - \mathrm{i}zA_{MX}^{(2)} - \frac{\mathrm{i}zV_1}{2M} C_{MYY} - \frac{\mathrm{i}z^2}{2} V_1 C_{MXX}) \Big] E(M)\mathrm{e}^{Mz} \\ & + \epsilon^2 A(M+N)E(M+N)\mathrm{e}^{(M+N)z} + \epsilon^2 A(M-N)E(M-N)\mathrm{e}^{(M-N)z} \\ & + \epsilon^2 A(2N)E(2N)\mathrm{e}^{2Nz} + \epsilon^2 A(2M)E(2M)\mathrm{e}^{2Mz} + (\text{c.c.}),\end{aligned} \tag{2.5a}$$

$$\begin{aligned}\phi_2 = \epsilon \Big[& -\mathrm{i}V_2 C_N + \epsilon(B_N^{(2)} + zV_2 C_{NX}) \\ & + \epsilon^2 (B_N^{(3)} + \mathrm{i}zB_{NX}^{(2)} - \frac{\mathrm{i}zV_2}{2N} C_{NYY} + \frac{\mathrm{i}z^2}{2} V_2 C_{NXX} \Big] E(N)\mathrm{e}^{-Nz} \\ & + \epsilon \Big[-\mathrm{i}V_2 C_M + \epsilon(B_M^{(2)} + zV_2 C_{MX}) \\ & + \epsilon^2 (B_M^{(3)} + \mathrm{i}zB_{MX}^{(2)} - \frac{\mathrm{i}zV_2}{2M} C_{MYY} + \frac{\mathrm{i}z^2}{2} V_2 C_{MXX} \Big] E(M)\mathrm{e}^{-Mz} \\ & + \epsilon^2 B(M+N)E(M+N)\mathrm{e}^{-(M+N)z} + \epsilon^2 B(M-N)E(M-N)\mathrm{e}^{-(M-N)z} \\ & + \epsilon^2 B(2N)E(2N)\mathrm{e}^{-2Nz} + \epsilon^2 B(2M)E(2M)\mathrm{e}^{-2Mz} + (\text{c.c.}),\end{aligned} \tag{2.5b}$$

$$\begin{aligned}\eta = {} & \epsilon(C_N + \epsilon C_N^{(2)} + \epsilon^2 C_N^{(3)})E(N) + \epsilon(C_M + \epsilon C_M^{(2)} + \epsilon^2 C_M^{(3)})E(M) \\ & + \epsilon^2 C(M+N)E(M+N) + \epsilon^2 C(M-N)E(M-N) \\ & + \epsilon^2 C(2N)E(2N) + \epsilon^2 C(2M)E(2M) + (\text{c.c.})\end{aligned} \tag{2.5c}$$

In the expansions (2.5) the coefficients $A_i^{(j)}, B_i^{(j)}, C_i$ etc. are functions of the slow variables X, Y, T, T_1 only and (c.c.) stands for complex conjugate. We assume without loss of generality that $M > N$ and also we assume that $M \neq 2N$ or $3N$. The reason for this is that the expansions (2.5) take on slightly different forms in these cases, which we hope to consider elsewhere. The next step is the rather tedious one of substituting (2.5) into the boundary conditions (2.1e, f) and matching ascending powers of ϵ. This task is made somewhat easier if (2.1e, f) are expanded about $z = 0$. If this is done they become, to the relevant order and bearing in mind that the conditions (2.4) hold,

$$\eta_t - \phi_{jz} + (V_j + \omega)\eta_x - \eta\phi_{jzz} + \eta_x\phi_{jx} - \tfrac{1}{2}\eta^2\phi_{jzzz} + \eta\eta_x\phi_{jxz} = 0, \quad z = 0, \quad j = 1, 2 \tag{2.6a}$$

and

$$\begin{aligned}\rho\phi_{2t} - \phi_{1t} + \rho(V_2 + \omega)\phi_{2x} - (V_1 + \omega)\phi_{1x} - \frac{MN}{M+N}(V_1^2 + \rho V_2^2)\eta \\ + \frac{(V_1^2 + \rho V_2^2)}{M+N}(\eta_{xx} + \eta_{yy}) + \rho\eta\phi_{2tz} - \eta\phi_{1tz}\end{aligned}$$

$$+ \rho(V_2+\omega)\eta\phi_{2xz} - (V_1+\omega)\eta\phi_{1xz} + \tfrac{1}{2}\rho(\phi_{2x}^2+\phi_{2z}^2)$$
$$- \tfrac{1}{2}(\phi_{1x}^2+\phi_{1z}^2) + \tfrac{1}{2}\rho\eta^2\phi_{2tzz} + \rho\eta\phi_{2x}\phi_{2xz}$$
$$+ \rho\eta\phi_{2z}\phi_{2zz} + \tfrac{1}{2}\rho(V_2+\omega)\eta^2\phi_{2xzz}$$
$$- \tfrac{1}{2}\eta^2\phi_{1tzz} - \eta\phi_{1x}\phi_{1xz} - \eta\phi_{1z}\phi_{1zz} - \tfrac{1}{2}(V_1+\omega)\eta^2\phi_{1xzz} - \frac{3(V_1^2+\rho V_2^2)}{2(M+N)}\eta_x^2\eta_{xx} = 0. \tag{2.6b}$$

These expansions should actually contain some terms involving y-derivatives. However since in the expansions (2.5) y only occurs as a slow variable they contribute nothing at the relevant order and consequently are omitted. It now turns out that the terms of order ϵ are already matched by choice of the leading-order terms in (2.5). When we come to match the terms of order $\epsilon^2E(2N)$, the kinematic conditions give

$$A(2N) - \mathrm{i}V_1C(2N) + \mathrm{i}NV_1C_N^2 = 0, \tag{2.7a}$$

$$B(2N) + \mathrm{i}V_2C(2N) + \mathrm{i}NV_2C_N^2 = 0, \tag{2.7b}$$

while Bernoulli's gives

$$2\mathrm{i}V_1A(2N) - 2\mathrm{i}\rho V_2B(2N) + \frac{(M+4N)}{(M+N)}(V_1^2+\rho V_2^2)C(2N) + N(\rho V_2^2 - V_1^2)C_N^2 = 0. \tag{2.7c}$$

Solving yields

$$A(2N) = \frac{\mathrm{i}V_1N(3NV_1^2+\rho(N-2M)V_2^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_N^2, \tag{2.8a}$$

$$B(2N) = \frac{\mathrm{i}V_2N(3\rho NV_2^2+(N-2M)V_1^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_N^2, \tag{2.8b}$$

$$C(2N) = \frac{N(M+N)(V_1^2-\rho V_2^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_N^2. \tag{2.8c}$$

Consideration of other terms at the quadratic level leads us to

$$A(2M) = \frac{\mathrm{i}V_1M(3MV_1^2+(M-2N)\rho V_2^2)}{(N-2M)(V_1^2+\rho V_2^2)}C_M^2, \tag{2.9a}$$

$$B(2M) = \frac{\mathrm{i}V_2M(3\rho MV_2^2+(M-2N)V_1^2)}{(N-2M)(V_1^2+\rho V_2^2)}C_M^2, \tag{2.9b}$$

$$C(2M) = \frac{M(M+N)(V_1^2-\rho V_2^2)}{(N-2M)(V_1^2+\rho V_2^2)}C_M^2, \tag{2.9c}$$

$$A(M+N) = \frac{\mathrm{i}V_1(M+N)(\rho V_2^2-3V_1^2)}{(V_1^2+\rho V_2^2)}C_MC_N, \tag{2.10a}$$

$$B(M+N) = \frac{\mathrm{i}V_2(M+N)(V_1^2-3\rho V_2^2)}{(V_1^2+\rho V_2^2)}C_MC_N, \tag{2.10b}$$

$$C(M+N) = \frac{2(M+N)(\rho V_2^2-V_1^2)}{(V_1^2+\rho V_2^2)}C_MC_N, \tag{2.10c}$$

$$A(M-N) = \frac{\mathrm{i}V_1(M+N)(MV_1^2+(4N-3M)\rho V_2^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_MC_N^*, \tag{2.11a}$$

$$B(M-N)=\frac{\mathrm{i}V_2(M+N)((4N-3M)V_1^2+M\rho V_2^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_M C_N^*, \tag{2.11b}$$

$$C(M-N)=\frac{2(M^2-N^2)(V_1^2-\rho V_2^2)}{(M-2N)(V_1^2+\rho V_2^2)}C_M C_N^*. \tag{2.11c}$$

(The asterisk stands for complex conjugate.) When the terms of the form $\epsilon^2 E(N)$ are considered, the kinematic conditions give

$$NA_N^{(2)}-\mathrm{i}NV_1C_N^{(2)}-C_{NT}-\omega C_{NX}=0, \tag{2.12a}$$

$$NB_N^{(2)}+\mathrm{i}NV_2C_N^{(2)}+C_{NT}+\omega C_{NX}=0, \tag{2.12b}$$

while Bernoulli's yields

$$\begin{aligned}\mathrm{i}NV_1A_N^{(2)}-\mathrm{i}\rho NV_2B_N^{(2)}+N(V_1^2+\rho V_2^2)C_N^{(2)}+\mathrm{i}(V_1+\rho V_2)C_{NT}\\ +\frac{\mathrm{i}((M-N)(V_1^2+\rho V_2^2)+\omega(M+N)(V_1+\rho V_2))}{(M+N)}C_{NX}=0.\end{aligned} \tag{2.12c}$$

Eliminating $A_N^{(2)}$ and $B_N^{(2)}$ between these three equations then leads to the result that

$$C_{NT}=s(N,M)C_{NX} \tag{2.13a}$$

where

$$s(N,M)=\frac{(N-M)(V_1^2+\rho V_2^2)-2\omega(M+N)(V_1+\rho V_2)}{2(M+N)(V_1+\rho V_2)} \tag{2.13b}$$

which in turn means that

$$A_N^{(2)}=\mathrm{i}V_1C_N^{(2)}+\frac{(N-M)(V_1^2+\rho V_2^2)}{2N(M+N)(V_1+\rho V_2)}C_{NX} \tag{2.14a}$$

and

$$B_N^{(2)}=-\mathrm{i}V_2C_N^{(2)}+\frac{(M-N)(V_1^2+\rho V_2^2)}{2N(M+N)(V_1+\rho V_2)}C_{NX}. \tag{2.14b}$$

(It should be noted that here and elsewhere the analysis breaks down when $V_1+\rho V_2$ is zero, in which case certain terms become singular. However, this simply corresponds to the onset of the Kelvin–Helmholtz instability, for it may easily be seen from (2.4*c*) that as $V_1+\rho V_2$ passes through zero, the roots of (2.4*c*) change from real to complex.) Consideration of the terms in $\epsilon^2 E(M)$ yields the analogous relations:

$$C_{MT}=s(M,N)C_{MX}, \tag{2.15}$$

$$A_M^{(2)}=\mathrm{i}V_1C_M^{(2)}+\frac{(M-N)(V_1^2+\rho V_2^2)}{2M(M+N)(V_1+\rho V_2)}C_{MX}, \tag{2.16a}$$

$$B_M^{(2)}=-\mathrm{i}V_2C_M^{(2)}+\frac{(N-M)(V_1^2+\rho V_2^2)}{2M(M+N)(V_1+\rho V_2)}C_{MX}. \tag{2.16b}$$

At the cubic order, we must now consider terms of the form $\epsilon^3 E(N)$. The kinematic terms yield

$$
\begin{aligned}
C_{NT_1} &+ C^{(2)}_{NT} + (V_1+\omega)C^{(2)}_{NX} + \mathrm{i}NV_1C^{(3)}_N - NA^{(3)}_N \\
&+ \mathrm{i}A^{(2)}_{NX} + (\mathrm{i}V_1/2N)C_{NYY} - 2N^2A(2N)C^*_N \\
&- \mathrm{i}N^2V_1C(2N)C^*_N - N(M+N)A(M+N)C^*_M \\
&- \mathrm{i}MNV_1C(M+N)C^*_M + N(M-N)A^*(M-N)C_M \\
&- \mathrm{i}MNV_1C^*(M-N)C_M - \tfrac{3}{2}\mathrm{i}N^3V_1|C_N|^2C_N - \mathrm{i}N(N^2+2M^2)V_1|C_M|^2C_N = 0
\end{aligned} \tag{2.17a}
$$

and

$$
\begin{aligned}
C_{NT_1} &+ C^{(2)}_{NT} + (V_2+\omega)C^{(2)}_{NX} + \mathrm{i}NV_2C^{(3)}_N \\
&+ NB^{(3)}_N - \mathrm{i}B^{(2)}_{NX} + (iV_2/2N)C_{NYY} - 2N^2B(2N)C^*_N \\
&+ \mathrm{i}N^2V_2C(2N)C^*_N - N(M+N)B(M+N)C^*_M \\
&+ \mathrm{i}MNV_2C(M+N)C^*_M + N(M-N)B^*(M-N)C_M \\
&+ \mathrm{i}MNV_2C^*(M-N)C_M - \tfrac{3}{2}\mathrm{i}N^3V_2|C_N|^2C_N - \mathrm{i}N(N^2+2M^2)V_2|C_M|^2C_N = 0,
\end{aligned} \tag{2.17b}
$$

while the Bernoulli condition yields

$$
\begin{aligned}
\mathrm{i}\rho NV_2B^{(3)}_N &- \mathrm{i}NV_1A^{(3)}_N - N(V_1^2+\rho V_2^2)C^{(3)}_N \\
&- \mathrm{i}(V_1+\rho V_2)C_{NT_1} + \rho(V_2+\omega)B^{(2)}_{NX} - (V_1+\omega)A^{(2)}_{NX} \\
&+ \rho B^{(2)}_{NT} - A^{(2)}_{NT} + \frac{2\mathrm{i}N}{M+N}(V_1^2+\rho V_2^2)C^{(2)}_{NX} \\
&+ \frac{(V_1^2+\rho V_2^2)}{M+N}C_{NXX} + \frac{(V_1^2+\rho V_2^2)}{M+N}C_{NYY} + N^2(V_1^2-\rho V_2^2)C(2N)C^*_N \\
&+ \mathrm{i}(M^2-N^2)V_1A(M+N)C^*_M + \mathrm{i}\rho(M^2-N^2)V_2B(M+N)C^*_M \\
&+ M^2(V_1^2-\rho V_2^2)C(M+N)C^*_M \\
&+ \mathrm{i}(N^2-M^2)V_1A^*(M-N)C_M + \mathrm{i}\rho(N^2-M^2)V_2B^*(M-N)C_M \\
&+ M^2(V_1^2-\rho V_2^2)C^*(M-N)C_M \\
&- \tfrac{5}{2}N^3(V_1^2+\rho V_2^2)|C_N|^2C_N + \frac{3}{2}\frac{N^4}{(M+N)}(V_1^2+\rho V_2^2)|C_N|^2C_N \\
&+ (N^3-2M^2N-2MN^2-2M^3)(V_1^2+\rho V_2^2)|C_M|^2C_N \\
&+ \frac{3M^2N^2}{M+N}(V_1^2+\rho V_2^2)|C_M|^2C_N = 0.
\end{aligned} \tag{2.17c}
$$

(These and certain other calculations were accomplished with the help of MATHEMATICA.) The next step in the procedure is to use (2.17*b*) to substitute for $A^{(3)}_N$ and $B^{(3)}_N$ in (2.17*c*).

The resulting equation is

$$
\begin{aligned}
&-2\mathrm{i}(V_1+\rho V_2)C_{NT_1} - \omega A^{(2)}_{NX} - \rho\omega B^{(2)}_{NX} \\
&+\frac{\mathrm{i}}{(M+N)}\{(N-M)(V_1^2+\rho V_2^2) - \omega(M+N)(V_1+\rho V_2)\}C^{(2)}_{NX} \\
&-A^{(2)}_{NT} + \rho B^{(2)}_{NT} - \mathrm{i}(V_1+\rho V_2)C^{(2)}_{NT} + \frac{(V_1^2+\rho V_2^2)}{M+N}C_{NXX} + \frac{(M+3N)}{2N(M+N)}(V_1^2+\rho V_2^2)C_{NYY} \\
&+2\mathrm{i}N^2(V_1A(2N)+\rho V_2B(2N))C^*_N
\end{aligned}
$$

$$+\mathrm{i}M(M+N)(V_1A(M+N)+\rho V_2B(M+N))C_M^*$$
$$+M(M-N)(V_1^2-\rho V_2^2)C(M+N)C_M^*$$
$$+\mathrm{i}(N-M)(M+2N)(V_1A^*(M-N)+\rho V_2B^*(M-N))C_M$$
$$+M(M-N)(V_1^2-\rho V_2^2)C^*(M-N)C_M-4N^3(V_1^2+\rho V_2^2)|C_N|^2C_N$$
$$-M(2M^2+4MN+2N^2)(V_1^2+\rho V_2^2)|C_M|^2C_N$$
$$+\frac{3N^2}{2(M+N)}(V_1^2+\rho V_2^2)(N^2|C_N|^2C_N+2M^2|C_M|^2C_N)=0. \tag{2.18}$$

Now we can use (2.13) and (2.14) to simplify the linear terms in (2.18). They become

$$-2\mathrm{i}(V_1+\rho V_2)C_{NT_1}-2\mathrm{i}(V_1+\rho V_2)C_{NT}^{(2)}$$
$$+\frac{\mathrm{i}}{M+N}\{(N-M)(V_1^2+\rho V_2^2)-2\omega(M+N)(V_1+\rho V_2)\}C_{NX}^{(2)}$$
$$+\left\{\frac{(V_1^2+\rho V_2^2)}{M+N}-\frac{(1+\rho)(M-N)^2(V_1^2+\rho V_2^2)^2}{4N(M+N)^2(V_1+\rho V_2)^2}\right\}C_{NXX}$$
$$+\frac{(M+3N)(V_1^2+\rho V_2^2)}{2N(M+N)}C_{NYY}, \tag{2.19}$$

where we have used (2.13) and its derivatives to write C_{NXT} and C_{NTT} in terms of C_{NXX}. If we now look at (2.14*a*), we see that $A_N^{(2)}-\mathrm{i}V_1C_N^{(2)}$ is a constant multiple of C_{NX}. Hence it would not seem unreasonable to make the *ansatz* that $C_N^{(2)}$ and $A_N^{(2)}$ each separately satisfy the relationship (2.13*a*) and as a consequence the terms involving $C_N^{(2)}$ vanish from (2.19). This means that the evolution equations, which will be presented shortly, take on a somewhat simpler form. It would be possible to proceed without this assumption at the expense of a somewhat more complicated set of evolution equations. However, this would not really lead to any great generalization for reasons which are discussed at the end of §3. The nonlinear terms in (2.18) can be simplified by use of (2.8)–(2.11) whereby an equation for C_N and C_M is obtained. Clearly an analogous consideration of the terms of the form $\epsilon^3E(M)$ yields a similar equation. Both equations can be simplified by introducing a new parameter V defined as V_2/V_1 and scaling T_1 to T_1/V_1. The final result of these calculations is the system

$$\mathrm{i}C_{MT_1}+u(M,N)C_{MXX}+v(M,N)C_{MYY}+p(M,N)|C_M|^2C_M+q(M,N)|C_N|^2C_M=0, \tag{2.20a}$$

$$\mathrm{i}C_{NT_1}+u(N,M)C_{NXX}+v(N,M)C_{NYY}+p(N,M)|C_N|^2C_N+q(M,N)|C_M|^2C_N=0, \tag{2.20b}$$

where the coefficients are given by

$$u(M,N)=\frac{(1+\rho V^2)\{(N^2-6MN-3M^2+\rho(M-N)^2)(1+\rho V^2)-8\rho M(M+N)V\}}{8M(M+N)^2(1+\rho V)^3}, \tag{2.21a}$$

$$v(M,N)=\frac{-(3M+N)(1+\rho V^2)}{4M(M+N)(1+\rho V)}, \tag{2.21b}$$

$$p(M,N)=\frac{M^3\{(8N^2+MN+2M^2)(1+\rho^2V^4)-6M(5N+2M)\rho V^2\}}{4(N-2M)(M+N)(1+\rho V^2)(1+\rho V)}, \tag{2.21c}$$

$$q(M,N)=\frac{N^2\{M(2N^2+10MN-M^2)(1+\rho^2V^4)-2(8N^3+14MN^2-2M^2N+M^3)\rho V^2\}}{2(M-2N)(M+N)(1+\rho V^2)(1+\rho V)}. \tag{2.21d}$$

The pair of coupled nonlinear partial differential equations (2.20) model, up to cubic order, the evolution of a capillary–gravity wavetrain occurring on the interface of two semi-infinite stratified fluids and caused by the interaction of the Mth and Nth harmonics of a fundamental mode. They are not valid when $M = 2N$ (when two of the coefficients become singular) or $M = 3N$. The reason for this latter exclusion is that the terms $2M, 2N, M \pm N$ are not all distinct and so the expansions (2.5) take a rather different form. Also some additional terms have to be adjoined to the evolution equations in this case, for instance (2.20*a*) would contain terms of the form A_N^3. It would, of course, be possible to take the analysis to a higher order. There are a number of reasons why we do not do this. One is that equations of the cubic nonlinear Schrödinger type have proved highly successful in predicting wave evolution over the last thirty years or so. Another is that it is generally thought that the lowest-order resonant interaction is the one which will dominate (see Hammack & Henderson 1993). Finally, on purely practical grounds, an extraordinary amount of additional work would be needed, even to take the analysis to fourth order, although a few researchers have undertaken such studies, see Dysthe (1979), Jones (1994*b*), Trulsen & Dysthe (1996). There are four parameters in the equations: the mode numbers M and N; V which is a measure of the ratio of the velocities and ρ which is the ratio of the densities. The equations are a generalization of another coupled pair occurring in Jones (1993, 1994*a*) which model the evolution of the analogous waves in the situation when the upper fluid is absent. They reduce to these equations in the case $\rho V = 0$. The equations in Jones (1993, 1994*a*) are themselves generalizations of the single nonlinear Schrödinger equation used by many authors to model the motion in the non-resonant case (see for instance Zakharov 1968; Hasimoto & Ono 1972).

3. Nonlinear wavetrains

It is an easy exercise to verify that the system (2.20) admits the Stokes wave solutions:

$$C_N = A_0 \exp\{\mathrm{i}N(\ell A_0 X + \gamma A_0^2 T_1)\}, \tag{3.1a}$$

$$C_M = \pm\lambda^{1/2} A_0 \exp\{\mathrm{i}M(\ell A_0 X + \gamma A_0^2 T_1)\}, \tag{3.1b}$$

where ℓ and A_0 are arbitrary real constants. The quantity λ is given by

$$\lambda = \frac{Nq(M,N) - Mp(N,M) + \ell^2 MN(Nu(N,M) - Mu(M,N))}{Mq(M,N) - Np(M,N)} \tag{3.2a}$$

$$= \frac{(2M-N)(2N-M)}{Mr(M,N)} \left\{ N^2(M(10M+N)(1+\rho^2V^4) + (4M^2 - 30MN - 16N^2)\rho V^2) - \frac{2\ell^2 M(M-N)(1+\rho V^2)^2(1+2\rho V+\rho V^2)}{(1+\rho V)^2} \right\} \tag{3.2b}$$

where

$$\begin{aligned} r(M,N) = {} & (2 - 12\rho V^2 + 2\rho^2 V^4)M^5 - (7 + 14\rho V^2 + 7\rho^2 V^4)M^4 N \\ & +(48 + 80\rho V^2 + 48\rho^2 V^4)M^3 N^2 - (28 + 120\rho V^2 + 28\rho^2 V^4)M^2 N^3 \\ & -(4 + 8\rho V^2 + 4\rho^2 V^4)MN^4 + 32\rho V^2 N^5. \end{aligned} \tag{3.2c}$$

The other quantity γ is given by

$$\gamma = \frac{q(M,N)^2 - p(N,M)p(M,N) + \ell^2(N^2u(N,M)p(M,N) - M^2u(M,N)q(M,N))}{Mq(M,N) - Np(M,N)}. \tag{3.3}$$

(If λ is negative we define $\lambda^{1/2}$ in (3.1*b*) to have positive imaginary part. The algebraic formula for γ is omitted because it is somewhat complicated and not of especial importance.) Because of the $\pm$ sign in (3.1*b*) we see that for given values of the parameters, two solutions are possible for given values of the parameters. Physically the one with the minus sign may be regarded as the one with the plus sign modified by a 'phase lag' of $\pi/(\gamma A_0^2 M)$. The corresponding expression (2.5*c*) for η, the sinusoidal wave profile, may then be obtained using (2.7)–(2.11). We remark that the expansion is not uniquely determined since the coefficients $C_N^{(2)}$ and $C_M^{(2)}$ occurring in η are arbitrary. We shall set them equal to zero. (For a discussion of this point see the end of the section.) A calculation reveals that when λ is positive, the corresponding wave profile is, up to second order,

$$\begin{aligned}\eta = {}& \epsilon\cos N\chi \pm \epsilon\lambda^{1/2}\cos M\chi \pm \epsilon^2\lambda^{1/2}(M+N)\frac{(\rho V^2-1)}{(\rho V^2+1)}\cos(M+N)\chi \\ & \pm\epsilon^2\lambda^{1/2}\frac{(M^2-N^2)(1-\rho V^2)}{(M-2N)(1+\rho V^2)}\cos(M-N)\chi \\ & +\frac{\epsilon^2 N(M+N)(1-\rho V^2)}{(M-2N)(1+\rho V^2)}\cos 2N\chi + \epsilon^2\lambda\frac{M(M+N)(1-\rho V^2)}{2(2M-N)(1+\rho V^2)}\cos 2M\chi,\end{aligned} \tag{3.4a}$$

while when λ is negative it is

$$\begin{aligned}\eta = {}& \epsilon\cos N\chi \mp \epsilon|\lambda|^{1/2}\sin M\chi \pm \epsilon^2|\lambda|^{1/2}(M+N)\frac{(1-\rho V^2)}{(1+\rho V^2)}\sin(M+N)\chi \\ & \mp\epsilon^2|\lambda|^{1/2}\frac{(M^2-N^2)(1-\rho V^2)}{(M-2N)(1+\rho V^2)}\sin(M-N)\chi + \epsilon^2\frac{N(M+N)(1-\rho V^2)}{2(M-2N)(1+\rho V^2)}\cos 2N\chi \\ & +\epsilon^2\lambda\frac{M(M+N)(1-\rho V^2)}{2(N-2M)(1+\rho V^2)}\cos 2M\chi.\end{aligned} \tag{3.4b}$$

(In these expansions $\chi = x - \omega t + \frac{1}{2}\ell X + \frac{1}{4}\gamma T_1$ and we have normalized by setting $A_0 = \frac{1}{2}$.) Let us make some observations concerning the types of wave profiles which may arise. First note that it is sufficient to consider (3.4) when $t = 0$, for fixing t at some different non-zero value merely amounts to a horizontal translation of the origin but does not alter the appearance of the wave. We then see that when λ is positive and so the wave profile is given by (3.4*a*), the wavetrain is symmetric in the sense that $\eta(x) = \eta(-x)$ while when λ is negative and the profile is given by (3.4*b*), no such symmetry is present. It is of interest to see how the values of the parameters determine the shape of the waves. We shall first consider the situation when the parameter ℓ is set equal to zero. Observe first that this means that λ may now be regarded as a function of two variables: ρV^2 and the ratio $R = M/N$. Of course we must have $\rho V^2 > 0$ and $R > 1$. Then the term in the curly brackets in (3.2*b*) may be regarded as a quadratic in ρV^2 whose coefficients are functions of R:

$$(10R+1)R(\rho V^2)^2 + (4R^2 - 30R - 16)\rho V^2 + (10R+1)R. \tag{3.5}$$

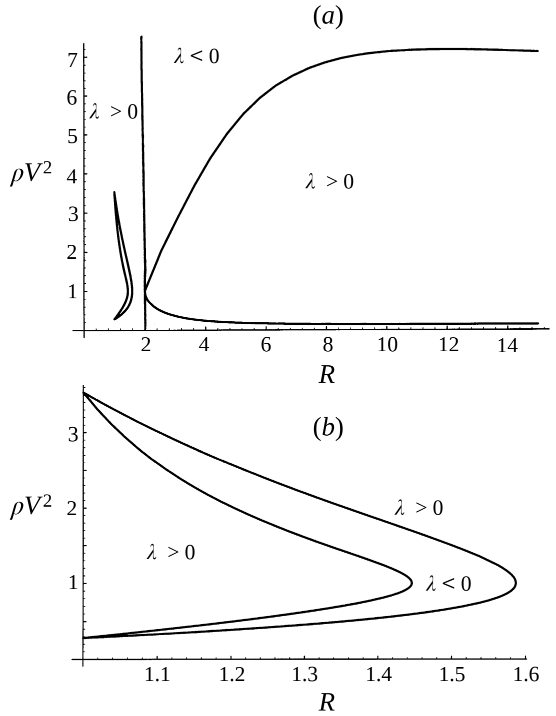

FIGURE 2. The sign of λ ((3.2) when $\ell = 0$) in the $(R, \rho V^2)$-plane where $R = M/N$; (b) is an enlargement of the left side of (a).

If we set this equal to zero and solve for ρV^2, we discover that it has the relatively simple solution

$$\rho V^2 = \frac{(2R+1)(8-R) \pm 4(1+R)(4+7R-6R^2)^{1/2}}{(10R+1)R}. \tag{3.6}$$

The term under the square root is easily found to vanish when $R = \frac{1}{12}(7+\sqrt{145}) \sim 1.586$ (the only relevant value for our purposes). Hence if $R > 1.586$ we conclude that (3.5) is always positive, while if $1 < R < 1.586$ then (3.5) has two real zeros both of which, a calculation shows, are always positive. Hence in this latter case, we can arrange for (3.5) to have either sign by judicious choice of ρV^2.

The denominator $r(M, N)$ in (3.2b) may be similarly dealt with by writing it as

$$(1+(\rho V^2)^2)(2R^5-7R^4+48R^3-28R^2-4R)+\rho V^2(32-8R-120R^2+80R^3-14R^4-12R^5), \tag{3.7}$$

again being regarded as a quadratic in ρV^2. Solving this leads us to

$$\rho V^2 = \{R(2R^4-7R^3+48R^2-28R-4)\}^{-1}\{(6R^5+7R^4-40R^3+60R^2+4R-16) \\ \pm 4(R+1)[(R-2)(2R-1)(R^3-2R^2+4R-2)(R^3+6R^2-8R-4)]^{1/2}\}. \tag{3.8}$$

A calculation now shows that the discriminant vanishes when $R = 2$ and $R \sim 1.446$ (for $R > 1$). We therefore conclude that for all values of R not between 1.446 and 2 the expression (3.7) vanishes for two values of ρV^2 both of which, a calculation shows, are always positive. The zeros and signs of (3.5) and (3.7), and hence of λ, in the $(R, \rho V^2)$ plane are depicted in figure 2.

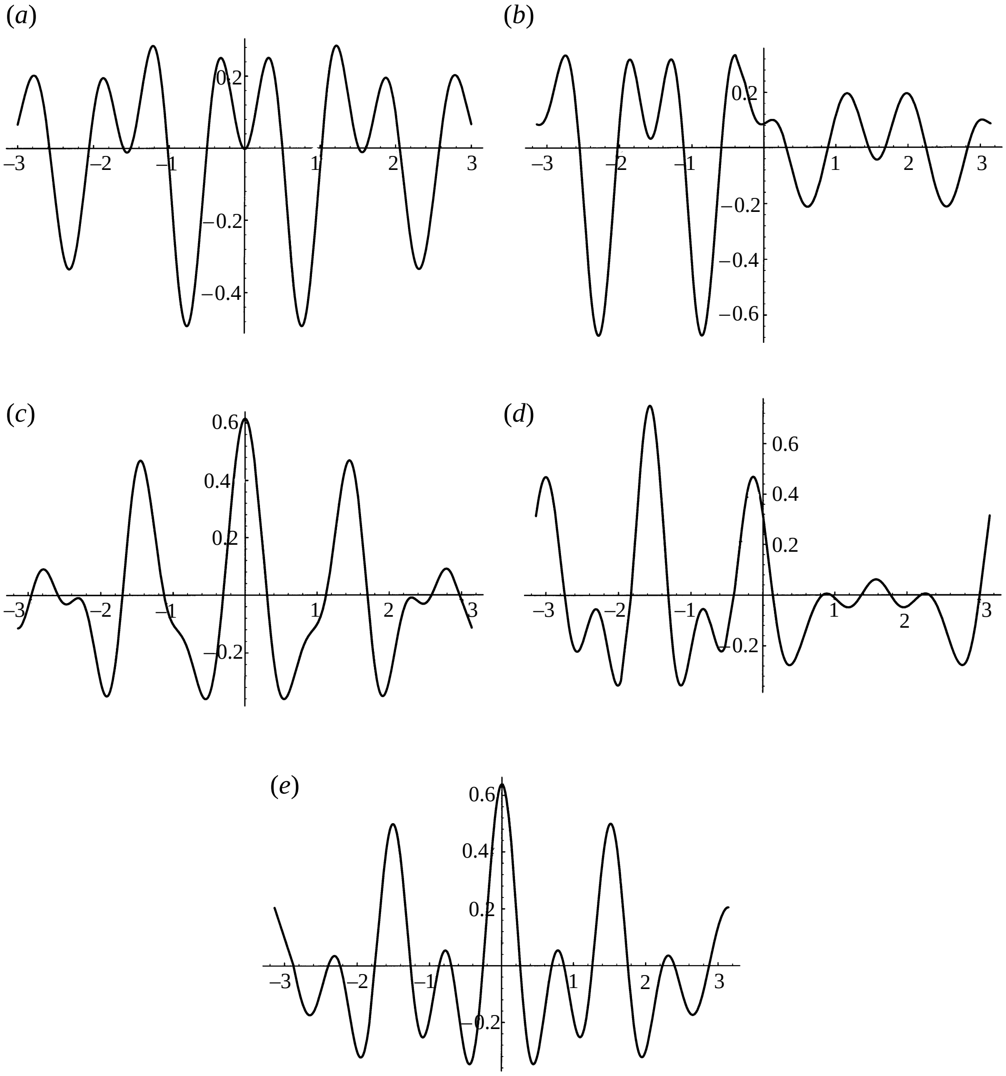

FIGURE 3. Some of the wave profiles for selected values of the parameters. The symmetric profiles are given by (3.4*a*) and the asymmetric ones by (3.4*b*). In all cases the top sign has been taken and we have set ϵ equal to 0.2. Here we have held $M = 5$ and $N = 4$ fixed and allowed ρV^2 to vary. Each choice corresponds to a different region in figure 2. (*a*) $\rho V^2 = 0.4$, (*b*) 0.5, (*c*) 1.5, (*d*) 2.0, (*e*) 3.0.

We conclude that the sign of λ needs to be considered in the four regions: $1 < R < 1.446, 1.446 < R < 1.586, 1.586 < R < 2, R > 2$. In all but the third of these regions λ can be positive or negative according to the value of ρV^2 and hence both symmetric and asymmetric waves may arise. In the third region λ is always positive so only symmetric waves can arise. Some of the wave profiles for values of the parameters in the different regions are depicted in figures 3 and 4.

Finally we remark on the significance of the parameter ℓ. It is not hard to see from (3.2*b*) that if that part of λ independent of ℓ and that part multiplying ℓ^2 have the same sign, then increasing ℓ^2 from zero has no effect on the symmetries of the waves. If, on the other hand, they have different signs, then increasing ℓ^2 will lead to a reversal in the symmetries of the waves. We conclude with some examples to show that all four sign

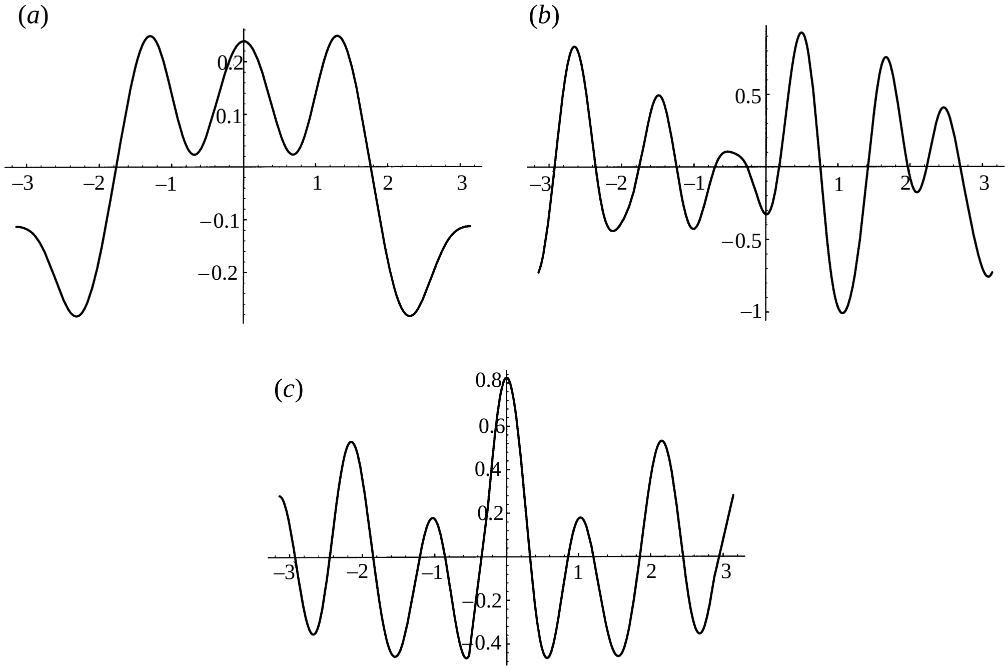

FIGURE 4. Wave profiles when ρV^2 is held fixed at 2.5 and R is allowed to vary. As before we have taken the top sign and $\epsilon = 0.2$. (*a*) $M = 4,\ N = 1$; (*b*) $M = 7,\ N = 3$; (*c*) $M = 5,\ N = 3$.

choices can occur: if $M = 7$ and $N = 5$, then for $(V, \rho) = (1, 0.6), (2, 0.5), (1.3, 0.5)$ the values of λ are $-0.169 - 0.0138\ell^2, 0.0735 - 0.00792\ell^2$ and $4.051 + 0.112\ell^2$ respectively, while if $M = 4, N = 1, V = 0.1, \rho = 0.8$ then $\lambda = -0.207 + 0.0309\ell^2$.

We conclude this section with some observations about generalizations of the evolution equations (2.20). Suppose, in particular, that we had not made the *ansatz* that $C_N^{(2)}$ satisfies (2.13*a*). This would mean that equation (2.20*b*) would contain additional terms proportional to $C_{NT}^{(2)} - s(N, M)C_{NX}^{(2)}$. (These and subsequent remarks apply *mutatis mutandis* to $C_M^{(2)}$ and (2.20*a*).) One consequence of this would be that (2.20) would no longer be a closed system but would now consist of two equations for four unknowns. (Although this would not be a terribly serious consequence, and note that even under our assumptions, $C_N^{(2)}$ and $C_M^{(2)}$ are not uniquely determined.) The simplest way of dealing with these modified evolution equations is then to take C_M and C_N as in (3.1) and to take $C_M^{(2)}$ and $C_N^{(2)}$ to be constants. Indeed, we see from (3.4) that the wave profile is only determined up to $O(\epsilon^2)$, so at this order any slow variation in $C_N^{(2)}$ and $C_M^{(2)}$ is negligible. Then since physically we are concerned with waves arising from an interaction between the Mth and Nth modes, the most natural choice for the constants is zero. Any non-zero choice would just be superimposing a 'free wave' of mode M or N on the flow which is mathematically an $O(\epsilon^2)$ solution of the linearized problem and would not interact at all at this level. Indeed some authors insist from the outset that in the perturbation expansion of η, the only coefficient of the Mth and Nth modes is the leading one of order ϵ (see Nayfeh 1970). In any case, it can easily be seen from the expansions (2.5*c*) and (3.4) of the wave profiles that the values of $C_M^{(2)}$ and $C_N^{(2)}$ have no qualitative effect on the form of the waves and it will become clear in §4 that neither do they affect their stability.

4. Stability considerations

In this Section we investigate the stability of the wavetrains found in §3. Because of the complexity of the algebra we confine ourselves to the solutions (3.1) for which $\ell = 0$. The first step is to make perturbations to the wave amplitudes C_N and C_M as follows:

$$C_N = A_0(1+\alpha)\exp\{\mathrm{i}N\gamma A_0^2 T_1 + \mathrm{i}\theta\}, \tag{4.1a}$$

$$C_M = \pm\lambda^{1/2} A_0(1+\beta)\exp\{\mathrm{i}M\gamma A_0^2 T_1 + \mathrm{i}\psi\}. \tag{4.1b}$$

Then substituting (4.1) into the evolution equations (2.20), linearizing and taking real and imaginary parts, we obtain

$$\theta_{T_1} - u(N,M)\alpha_{XX} - v(N,M)\alpha_{YY} - 2p(N,M)A_0^2\alpha - 2\lambda q(M,N)A_0^2\beta = 0, \tag{4.2a}$$

$$\alpha_{T_1} + u(N,M)\theta_{XX} + v(N,M)\theta_{YY} = 0, \tag{4.2b}$$

$$\psi_{T_1} - u(M,N)\beta_{XX} - v(M,N)\beta_{YY} - 2q(M,N)A_0^2\alpha - 2\lambda p(M,N)A_0^2\beta = 0, \tag{4.2c}$$

$$\beta_{T_1} + u(M,N)\psi_{XX} + v(M,N)\psi_{YY} = 0. \tag{4.2d}$$

We shall assume plane wave perturbations, so that

$$\begin{pmatrix}\alpha\\ \theta\\ \beta\\ \psi\end{pmatrix} = \begin{pmatrix}\overline{\alpha}\\ \overline{\theta}\\ \overline{\beta}\\ \overline{\psi}\end{pmatrix}\exp\{\mathrm{i}(\delta X + \eta Y) - \mathrm{i}\kappa T_1\}. \tag{4.3}$$

Then substituting into (4.2) we obtain a set of equations which can only be consistent if the following determinant vanishes:

$$\begin{vmatrix} \kappa, & P_1, & 0, & 0\\ P_1 - 2p(N,M), & \kappa, & -2\lambda q(M,N), & 0\\ 0, & 0, & \kappa, & P_2\\ -2q(M,N), & 0, & P_2 - 2\lambda p(M,N), & \kappa \end{vmatrix}. \tag{4.4}$$

In (4.4), the quantities P_1 and P_2 are defined as

$$P_1 = u(N,M)\delta^2 + v(N,M)\eta^2, \qquad P_2 = u(M,N)\delta^2 + v(M,N)\eta^2, \tag{4.5}$$

and A_0 has been eliminated by the scalings

$$\kappa \to A_0^2\kappa, \qquad X \to A_0 X, \qquad Y \to A_0 Y.$$

On expanding (4.4), we obtain the following quartic equation for κ:

$$\kappa^4 + C\kappa^2 + D = 0, \tag{4.6}$$

$$C = -P_1^2 - P_2^2 + 2(p(N,M)P_1 + \lambda p(M,N)P_2) \tag{4.7a}$$

$$D = P_1P_2[(P_1 - 2p(N,M))(P_2 - 2\lambda p(M,N)) - 4\lambda q^2(M,N)]. \tag{4.7b}$$

In addition we shall introduce the quantity $\Delta = C^2 - 4D$ and a calculation yields that

$$\Delta = \{P_1(P_1 - 2p(N,M)) - P_2(P_2 - 2\lambda p(M,N))\}^2 + 16\lambda q^2(M,N)P_1P_2. \tag{4.7c}$$

Clearly the waves will only be stable if all the roots of (4.6) are real and it is a standard exercise to determine that this only happens if either $C \leqslant 0$ and $D = 0$ or $C \leqslant 0, D > 0$ and $\Delta \geqslant 0$.

Because there are four parameters involved in the problem: M, N, V and ρ, it is not easy to give any very general results on the stability of the waves. However, we first make some general observations and then determine the regions of stability and instability for some particular cases. First note that for large values of the wavenumbers δ and η of the perturbations we must have, in general $C < 0, D > 0$ and $\Delta > 0$ and hence stability is assured. This would seem unsurprising since a perturbation with wavenumbers far from the main flow would not be expected to lead to instability. It may be seen from (4.7) that the quantities which have an important influence on the signs and zeros of C, D and Δ are P_1, P_2 and λ. The zeros of λ have already been discussed at the end of §3. An inspection of P_2 shows that for the quantity to have zeros in the (δ, η)-plane the factor

$$\{(1+\rho V^2)(\rho-3)-8\rho V\}R^2-2\{(1+\rho V^2)(3+\rho)+4\rho V\}R+(1+\rho)(1+\rho V^2), \quad (4.8)$$

occurring in the numerator of $u(M, N)$ must be positive (here $R = M/N$ as before). The coefficient of R^2 in (4.8) may clearly be arranged to be positive or negative according to the values of ρ and V. (Note, though, for it to be positive, V must be negative and so the undisturbed flows must be in opposition.) Then obviously when this term is positive, the quantity (4.8) is positive for sufficiently large R. Note however, that (4.8) cannot be positive for R close to unity. For if we set $R = 1$ in (4.8), it may be written as $-8\{(1+\rho V)^2+\rho V^2(1-\rho)\}$ which is strictly negative. (It can only be zero if $\rho = 1$ and $V = -1$, which case is specifically excluded.) We content ourselves with a single numerical example: if $V = -2$ and $\rho = 0.8$ then (4.8) equals

$$3.56R^2 - 19.12R + 7.56$$

which changes from negative to positive as R increases through 4.94.

Finally, we remark upon zeros of P_1. In a similar way, the condition for P_1 to have zeros is that the term

$$((R^2-6R-3)+\rho(R-1)^2)(1+\rho V^2)-8(R+1)\rho V \quad (4.9)$$

(occurring in the numerator of $u(N, M)$) be positive. However this is quite easy to arrange and indeed zeros of P_1 will tend to be the rule rather than the exception. First notice that if $R > 3+2\sqrt{3}$ (~ 6.46) then R^2-6R-3 is positive and it thus suffices to take V to be negative. However (4.9) can still be positive even if R is smaller than this critical value. For instance if we set $R = 2$ and $\rho = 0.9$, then (4.9) is positive if V lies between -4.51 and -0.247 and negative otherwise. On the other hand, if $R = 2$ and $\rho = 0.2$ then (4.9) is always negative. Note however, for R very close to unity, (4.9) is always negative for the same reasons as those given in the discussion of P_2.

We now present stability diagrams for some specific values of the parameters involved. We first consider the case when $\rho = 0.5$ and $V = 2$. Different values of M and N will be chosen to illustrate the different stability patterns which may occur.

(i) $M = 19, N = 18, \rho = 0.5, V = 2$

In this case a calculation reveals that P_1 and P_2 are both negative, while λ and hence Δ (by 4.7c) are positive. Further, C is negative while a calculation shows that the zeros of D form an ellipse-like curve depicted in figure 5(a). The quantity D is negative inside the ellipse and positive outside. The conclusion is therefore that the waves are unstable for perturbations with wavenumbers close to the origin and stable otherwise.

(ii) $M = 9, N = 7, \rho = 0.5, V = 2$

In this case P_1, P_2, C and λ are all negative, while D and Δ change sign. The zeros of these quantities, together with an indication of the stability regions, are depicted in figure 5(b). An inspection of this diagram shows that perturbations with wavenumbers sufficiently close to the underlying flow are always unstable, no matter

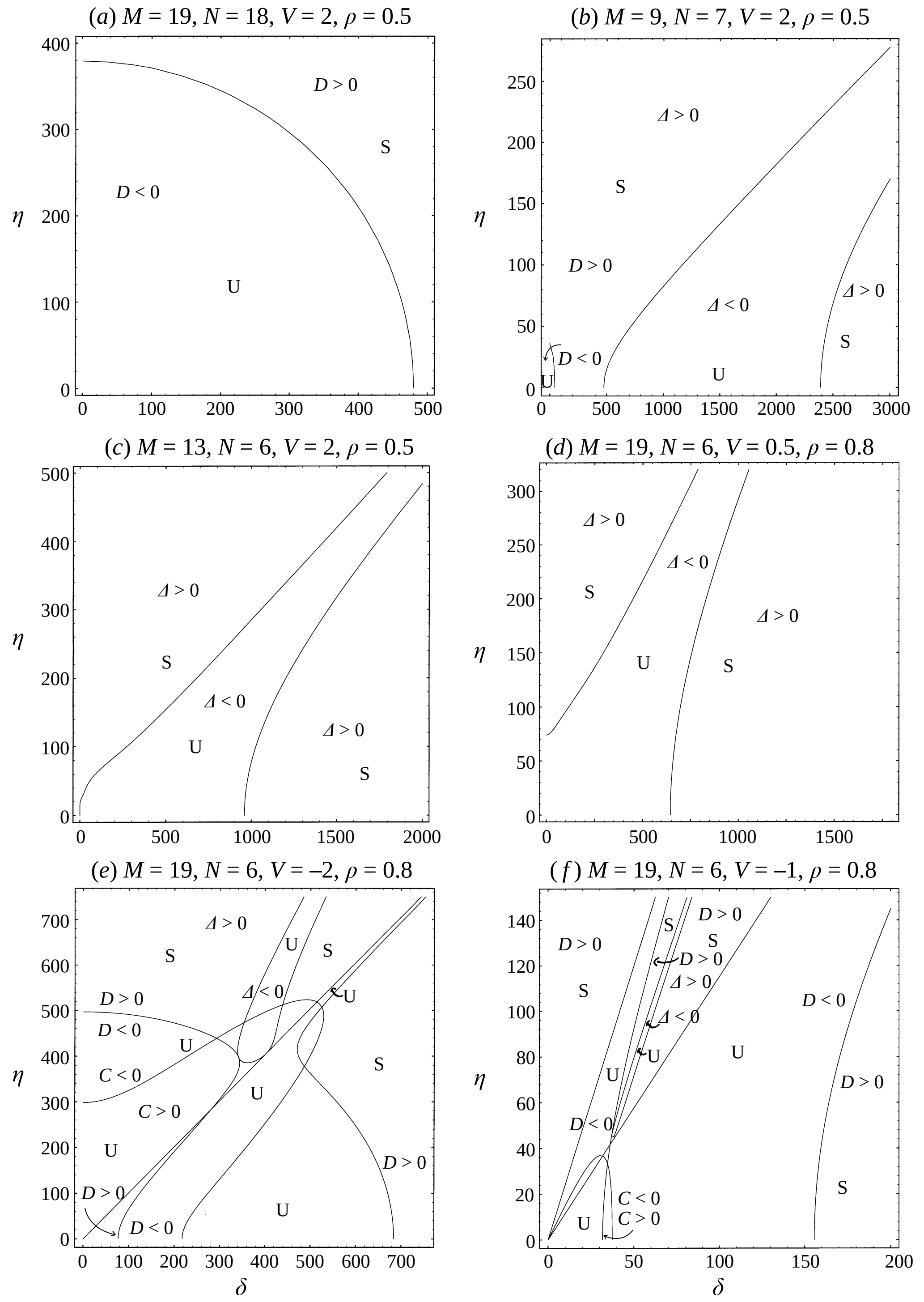

FIGURE 5. The stability regions for various values of the parameters. In all cases regions of stability are denoted by S and regions of instability by U.

what their direction may be. In addition, there is a second unbounded region of instability which includes some perturbations in the same direction as the flow, but none at right angles to it.

(iii) $M = 13, N = 6, \rho = 0.5, V = 2$

In this case P_1, P_2, C and λ are always negative, while D is positive. The only quantity to change sign is Δ, the zeros of which are depicted in figure 5(c), together with the stability implications. We see that the waves are always stable to perturbations transverse to the wave; whereas longitudinal perturbations are stable or unstable according to their wavenumber.

(iv) $M = 13, N = 4, \rho = 0.5, V = 2$

None of the relevant quantities changes sign in this case: P_1, P_2 and C are always negative; D, λ and hence Δ are always positive. The waves are therefore always stable.

We now present some stability results when M, N and ρ are held fixed, but for different values of V. We shall choose values of V to illustrate representative cases: one when neither P_1 nor P_2 changes sign; one when just one of them does, and one when they both do.

(v) $M = 19, N = 6, \rho = 0.8, V = 0.5$

Here P_1, P_2 and C are always negative, while D is always positive. The zeros of Δ, and associated stability implications are depicted in figure 5(d). It may be seen that the waves are unstable to perturbations with wavenumbers close to the origin, while if the wavenumbers are further away the waves are stable to perturbations which are purely longitudinal or transverse and the region of unstable oblique wavenumbers becomes narrower (but is unbounded).

(vi) $M = 19, N = 6, \rho = 0.8, V = -2$

In this case P_2 is always positive while all other quantities change sign and their zeros are depicted in figure 5(e), which also shows the regions of stability. Although figure 5(e) may appear somewhat complicated, a careful inspection reveals that, broadly speaking, the waves are unstable if the wavenumbers of the perturbations are close to the origin while they are stable if they are further away, with the exception of two unbounded regions of instability.

(vii) $M = 19, N = 6, \rho = 0.8, V = -1$

For these values of the parameters, all three of C, D and Δ change sign and the zero set of D is particularly complicated since P_1 and P_2 both change sign. The zeros of the various quantities and associated stability regions are depicted in figure 5(f). An inspection of this figure then shows that the waves are stable if the perturbations are purely transverse to the wave no matter what their wavenumber may be, while perturbations purely longitudinal to the wave are unstable if the wavenumbers are sufficiently small and stable otherwise. Further, oblique perturbations are stable or unstable according to the precise position of their wavenumbers in the plane. Perturbations with wavenumbers far from the origin tend to be stable, although there are three unbounded regions of instability.

5. Conclusions

We have carried out an investigation into the small-amplitude ripples which arise on the interface of two semi-infinite stratified fluids and which are caused by the interaction of the Mth and Nth harmonics of the fundamental mode. Our results are valid for all values of M and N except $M = 2N$ and $M = 3N$. A pair of coupled nonlinear partial differential equations which model, up to cubic order, the evolution of the waves has been derived. These are generalizations of the nonlinear

Schrödinger equation which has been extensively used to describe other types of wave evolution. The equations were solved and used to obtain formal series expansions for the wave profiles, in powers of the wave steepness. A very large number of wave profiles are possible and certainly, for given values of M and N, both symmetric and asymmetric waves may be observed. Four parameters are present in the problem: as well as M and N there are occurrences of V, which is a measure of the ratio of the fluid velocities, and ρ which is the density ratio of the fluids. There are two factors which have a strong influence on the character of the waves. One is whether M is greater or less than $2N$. The other is the value of the quantity ρV^2 which may be thought of as a sort of 'generalized velocity'. We note that if $\rho V^2 \sim 0$ then only one of ρ or V^2 need be near zero so this may be physically interpreted either as 'very light fluid on the top' or 'slow relative velocity'; but if $\rho V^2 \gg 1$ then, since $0 \leqslant \rho \leqslant 1$, the only realistic physical interpretation of this is 'very fast relative velocity'. However when V is very small or very large the difference between the velocities is large, so we might describe both these cases generically as 'strong shear'. (Note that when $V = 1$, the undisturbed flows are equal.) If we confine ourselves to Stokes-wave-type solutions which are functions of the slow time only, then we have shown that, if $M < 2N$, the waves are asymmetric for strong shear and symmetric for an intermediate range of value of ρV^2, whereas the conclusions are reversed if $M > 2N$. The results for $\rho V^2 \sim 0$ are the same as those contained in Jones (1994*a*) which dealt with capillary–gravity waves in an open channel and under constant atmospheric pressure. (The case $\rho = 0$ corresponds to surface waves, of course.) This is unsurprising, of course, since we would not expect a very light or slow-moving upper fluid to exert a dramatic effect. The reason for the observations in the other cases could be that a very fast-moving upper fluid might not be expected to have much effect on the lower one while a fluid which moves with only moderate speed could have a strong effect at the interface. Our investigations have shown that the 1:2 mode interaction is a particularly important one. Thus although not considered in this paper, this case would seem to merit further study, as would the 1:3 mode interaction. Indeed in the surface wave case these resonances are among the most observable, see Hammack & Henderson (1993). In fact some work on such problems has already been carried out by Bontozoglou & Hanratty (1990) and Christodoulides & Dias (1994) who used a Lagrangian formulation to consider stratified waves and Chossat & Dias (1995) who considered second-harmonic resonances in more general physical settings. It would be of great interest to extend their analyses to the resonant waves which have been the subject of this report.

Having discussed the types of waves which may occur, we proceeded to study their stability to plane wave perturbations. Naturally, the stability properties of the waves were shown to depend on the particular values of the parameters chosen. However, in general, perturbations with wavenumbers far from the main flow tend to have less of an effect on stability than perturbations with wavenumbers close to it. This is in accord with physical expectations since we would expect energy to be transferred more easily between modes whose wavenumbers are close to each other. In some cases the waves are stable to all perturbations with wavenumbers far from the main flow, but in others there are unbounded regions of instability, some of whose boundaries become asymptotically close as the wavenumbers of the perturbations move away from the origin, while others do not. Another observation gained from a perusal of the stability diagrams is that longitudinal perturbations tend, in the main, to be more unstable than transverse ones. Again this is physically reasonable, for one would expect an interaction to arise more easily from two wavetrains moving in the same direction rather than from two which are at right angles.

The results presented here are only valid in the case of small-amplitude waves. An interesting and obvious area of further work would be to try and obtain results valid for finite-amplitude waves, either by rigorous or numerical methods, rather as Zhang & Melville (1987) did for open channel capillary–gravity waves.

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REFERENCES

Aston, P. J. 1991 Local and global aspects of the $(1, n)$ mode interaction for capillary–gravity waves, *Physica* D **52**, 415–428.

Aston, P. J. 1993 Understanding the global solutions of the capillary–gravity wave problem. *Wave Motion* **17**, 113–141.

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wavetrains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

Bontozoglou, V. & Hanratty, T. J. 1990 Capillary–gravity Kelvin–Helmholtz waves close to resonance. *J. Fluid Mech.* **217**, 71–91.

Bridges, T. J. & Mielke, A. 1995 A proof of the Benjamin-Feir instability. *Arch. Rat. Mech. Anal.* **133**, 145–198.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Chen, B. & Saffman, P. G. 1979 Steady gravity–capillary waves on deep water – 1. Weakly nonlinear waves. *Stud. Appl. Maths* **60**, 183–210.

Chen, B. & Saffman, P. G. 1980 Steady capillary–gravity waves on deep water – II. Numerical results for finite amplitude. *Stud. Appl. Maths* **62**, 95–111.

Chossat, P. & Dias, F. 1995 The 1:2 resonance with $O(2)$ symmetry and its applications in hydrodynamics. *J. Nonlinear Sci.* **5**, 105–129.

Christodoulides, P. & Dias, F. 1994 Resonant capillary–gravity interfacial waves. *J. Fluid Mech.* **265**, 303–343.

Christodoulides, P. & Dias, F. 1995 Stability of capillary–gravity waves between two bounded fluids. *Phys. Fluids* **12**, 3013–3027.

Craik, A. D. D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

Davey, A. & Stewartson, K. 1974 On three dimensional packets of surface waves. *Proc. R. Soc. Lond.* A **338**, 101–110.

Djordjevic, V. D. & Redekopp, L. G. 1977 On two-dimensional packets of capillary–gravity waves. *J. Fluid Mech.* **79**, 703–714.

Dysthe, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

Hammack, J. L. & Henderson, D. M. 1993 Resonant interactions among surface water waves. *Ann. Rev. Fluid Mech.* **25**, 55–97.

Hasimoto, H. & Ono, H. 1972 Nonlinear modulation of gravity waves. *J. Phys. Soc. Japan* **33**, 805–811.

Jones, M. C. W. 1992 Nonlinear stability of resonant capillary–gravity waves. *Wave Motion* **15**, 267–283.

Jones, M. C. W. 1993 On coupled differential equations which model the evolution of interacting capillary–gravity wave modes and related questions of stability. *IMA J. App. Maths* **50**, 13–28.

Jones, M. 1994*a* Further results on the stability of symmetric and asymmetric resonant capillary–gravity waves. *Japan J. Ind. Appl. Maths* **11**, 465–483.

Jones, M. 1994*b* On the stability of finite-depth capillary–gravity waves by means of the fourth order evolution equation. *Can. Appl. Maths Q.* **2**, 325–348.

Jones, M. & Toland, J. 1986 Symmetry and the bifurcation of capillary–gravity waves. *Arch. Rat. Mech. Anal.* **96**, 29–53.

Miles, J. W. 1986*a* Weakly nonlinear Kelvin–Helmholtz waves. *J. Fluid Mech.* **172**, 513–529.

Miles, J. W. 1986*b* Weakly nonlinear waves in a stratified fluid: a variational formulation. *J. Fluid Mech.* **172**, 499–512.

Nayfeh, A. H. 1970 Finite amplitude surface waves in a liquid layer. *J. Fluid Mech.* **40**, 671–684.

Nayfeh, A. H. 1971 Third harmonic resonance in the interaction of capillary and gravity waves. *J. Fluid Mech.* **48**, 384–395.

Nayfeh, A. H. & Saric, W. S. 1972 Nonlinear waves in a Kelvin–Helmholtz flow. *J. Fluid Mech.* **55**, 311–327.

Pierson, W. J. & Fife, P. 1961 Some nonlinear properties of long-crested periodic waves with length near 2.44 centimetres. *J. Geophys. Res.* **66**, 163–179.

Schwartz, L. W. & Vanden-Broeck, J.-M. 1979 Numerical solution of the exact equations for capillary–gravity waves. *J. Fluid Mech.* **95**, 119–139.

Toland, J. F. & Jones, M. C. W. 1985 The bifurcation and secondary bifurcation of capillary–gravity waves. *Proc. R. Soc. Lond.* A **399**, 391–417.

Trulsen, K. & Dysthe, K. B. 1996 A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water. *Wave Motion.* (to appear).

Wilton, J. R. 1915 On ripples. *Phil. Mag.* (6) **29**, 688–700.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **2**, 190–194.

Zhang, J. & Melville, W. K. 1987 Three dimensional instabilities of nonlinear gravity–capillary waves. *J. Fluid Mech.* **174**, 187–208.